BGD COLLEGE ,KESAIBAHAL

Blended learning modules

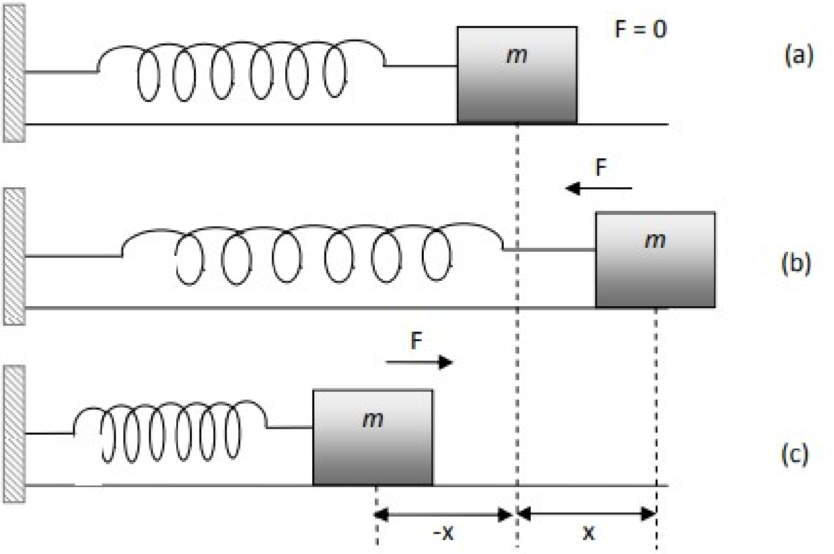
2nd Year 3rd SEM

Subject :- **GENERIC ELECTIVE (GE)**

SIMPLE HARMONIC MOTION

SHM can be defined in a number of ways:

1. If the force acting on the oscillating body is always in the direction opposite to the displacement of the body from the equilibrium or the mean position and its magnitude is proportional to the magnitude of displacement, the body is said to be executing SHM.
2. If the displacement vs. time curve of the oscillating body is sinusoidal in nature, the body is said to be executing SHM. This is another definition of SHM.
3. If the potential energy of the oscillating body is proportional to the square of its displacement with reference to the mean position, the body is said to be executing SHM. This is yet another definition of SHM.



if a spring of length L is stretched through a distance x by a force of magnitude F, the Young's modulus Y of the material of the wire is given by

$$Y = \frac{F/\alpha}{x/L} \tag{1.1}$$

Here α is the cross-sectional area of the wire. By rearranging the terms in equation (1.1), we can easily get the following form

$$F = \left(\frac{Y\alpha}{L}\right)x \tag{1.2}$$

$$F = -kx \tag{1.3}$$

Since, the restoring force, F is proportional to the displacement[1] and is opposite in sign to the displacement, the resulting motion is simple harmonic. Here k is called the spring constant or stiffness constant. The SI unit of k is Nm^{-1}.

1.5 DIFFERENTIAL EQUATION OF SHM

Let us now express equation (1.3) in the differential form by using Newton's second law of motion. From Newton's second law of motion, we know that force experienced by a body of mass m can be expressed as a function of acceleration,

$$F = ma = m\ddot{x}$$

Therefore, in a spring-mass system, the force can be written as

$$F = m\ddot{x} = -kx$$

Or we can say that

$$m\ddot{x} + kx = 0$$

$$or, \qquad \ddot{x} + \frac{k}{m}x = 0 \qquad (1.6)$$

(**Comment**: Either follow the double dot notation or d^2x/dt^2 notation for double differentiation. In later Units, d^2x/dt^2 notation has been used. For students, it will be better if we follow d^2x/dt^2 notation.)

The above equation is the differential equation of SHM. k is the force constant (for our case of spring-mass system, it is called the spring constant) and has dimensions $(MLT^{-2}/L) = MT^{-2}$.

Therefore, the dimension of k/m is T^{-2}, i.e. square of reciprocal of time. We can replace k/m by ω^2. Thus, the equation (1.6) takes the form

$$\ddot{x} + \omega^2 x = 0 \qquad (1.7)$$

We will find the physical meaning of ω, that it is actually the angular frequency that we already defined earlier, when we solve the differential equation (1.7).

1.5.1 Solution of the Differential Equation of SHM

The second time derivative of displacement ($\ddot{x}$) can be written as

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

Multiplying and dividing by dx in the numerator and the denominator, we get

$$\ddot{x} = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right)$$

We already know that $\dot{x}$ or dx/dt actually define the velocity$v$. Therefore, the above expression can take the following form

$$\ddot{x} = v\frac{d}{dx}(v)$$

Since,

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) = v\frac{dv}{dx}$$

We get

$$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right) \tag{1.8}$$

From (1.7) and (1.8), we get

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) + \omega^2 x = 0$$

$$or\ \ \frac{d}{dx}\left(\frac{v^2}{2} + \omega^2\frac{x^2}{2}\right) = 0$$

$$\therefore d(v^2 + \omega^2 x^2) = 0 \tag{1.9}$$

On integrating both the sides, we get

$$v^2 + \omega^2 x^2 = constant\ (C_1) \tag{1.10}$$

We already know that on the two extremes, when the magnitude of the displacement is equal to the amplitude ($x = \pm A$), the kinetic energy or the velocity is zero ($v = 0$). Using this boundary condition in equation (1.10), we can calculate the constant (C_1). Thus, C_1 is given by

$$(0)^2 + \omega^2(\pm A)^2 = C_1$$

$$or\ \ C_1 = \omega^2 A^2$$

Using this value in equation (1.10) and rearranging the terms, we get

$$v^2 = \omega^2(A^2 - x^2)$$

$$or\ \ v = \pm\omega\sqrt{(A^2 - x^2)} \tag{1.11}$$

The above relation is the expression for velocity of a particle executing SHM. We can see how the velocity has a maximum magnitude at $x = 0$ or in other words, the mean position. From (1.11), the maximum velocity is given by

$$|v|_{max} = \omega A \tag{1.12}$$

Now, we will determine the expression for the displacement of a particle executing SHM. From (1.11), we get

$$\frac{dx}{dt} = \pm\omega\sqrt{(A^2 - x^2)}$$

Rearranging the terms, we get

$$\pm\frac{dx}{\sqrt{(A^2 - x^2)}} = \omega dt$$

On integrating both the sides, we get corresponding to the (+) sign

$$\sin^{-1}\frac{x}{A} = \omega t + \delta_1$$

And, corresponding to the (−) sign

$$\cos^{-1}\frac{x}{A} = \omega t + \delta_2$$

where δ_1 and δ_2 are dimensionless constants.

Therefore, we can see that the SHM is defined by a sinusoidal curve

$$x(t) = A\sin(\omega t + \delta) \tag{1.13}$$

Depending on the value of constant δ and ωt the displacement from the equilibrium position and velocity of the SHM at any instant can be determined.

1.5.2 Angular Frequency of SHM

We know that the displacement $x(t)$ should return to its initial value after one time period T of the motion. Or

$$x(t) = x(t + T)$$

We also know from trigonometry that the sine or cosine function repeats itself when its argument has increased by $2\pi\ rad$. Thus,

$$\omega(t + T) = \omega t + 2\pi$$

Or, we get

$$\omega = \frac{2\pi}{T} = 2\pi\nu \tag{1.14}$$

The quantity ω is therefore, the angular frequency that we defined earlier. Its SI unit is $rad\ s^{-1}$.

From equation (1.6), we know that

$$\omega^2 = \frac{k}{m}$$

$$\therefore\ \omega = \sqrt{\frac{k}{m}} \tag{1.15}$$

damped harmonic motion

INTRODUCTION

When we consider an oscillator executing simple harmonic motion, it is assumed that the oscillations will continue for infinite time. In other words, we treat the whole system as an idealized frictionless system. But, in reality, this does not happen, and the oscillating object gradually loses its energy due to several factors. One such major factor is the frictional forces which diminish the amplitude of oscillation and the system ultimately comes to rest.

The decrease in amplitude caused by the dissipative forces is called damping and these oscillations with decreasing amplitude are called damped oscillations. In the present unit, we shall discuss the effects of damping on the oscillatory systems.

FRICTIONAL EFFECTS (DAMPING)

The oscillatory motion we considered so far, have been for ideal systems. It means that such oscillatory systems will oscillate indefinitely under the action of only one force – a linear restoring force. In the basic analysis of harmonic oscillators, we completely ignore the effect of frictional forces in it. But, in real situations, the oscillator is in a resistive medium like air, oil etc. In such conditions, part of the energy of the oscillator is spent in opposing frictional or viscous forces. At ordinary velocities, the opposing, resistive or damping force is to a first approximation, proportional to velocity and may be represented by

$$F = -\gamma v = -\gamma \frac{dx}{dt}$$

Where γ is a positive constant, called **damping coefficient** of the medium and may be termed as resistive force per unit velocity.

So, if there is no other force other than this resistive or damping force acting on the oscillating body of mass m, then Newton's second law of motion gives

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = -\gamma v$$

Or $$\frac{dv}{dt} + \frac{\gamma}{m} v = 0 \quad \text{------------------------------} (1)$$

Here, $\frac{m}{\gamma}$ is usually denoted by a constant, having dimensions of time and is called as relaxation time (τ).

Therefore,

$$\frac{dv}{dt} + \frac{1}{\tau} v = 0 \quad \text{------------------------------------} (2)$$

The constant $\frac{1}{\tau} = \frac{\gamma}{m}$, or the resistive force per unit mass per unit velocity, is often denoted by 2b, where b is called damping constant of the medium.

Now rewriting and integrating equation (2), we get

$$\int \frac{dv}{v} = -\frac{1}{\tau}\int dt$$

which gives

$$\ln v = -\frac{t}{\tau} + C$$

Where C is a constant of integration to be determined from the initial conditions.

Puting $t=0$, $v=v_0$ we get

$$\ln v_0 = C$$

therefore-

$$v = v_0 e^{-\frac{t}{\tau}} \quad \text{-------------------------------- (3)}$$

Above equation clearly shows that the velocity decreases exponentially with time, as shown by the curve in Fig 1 below.

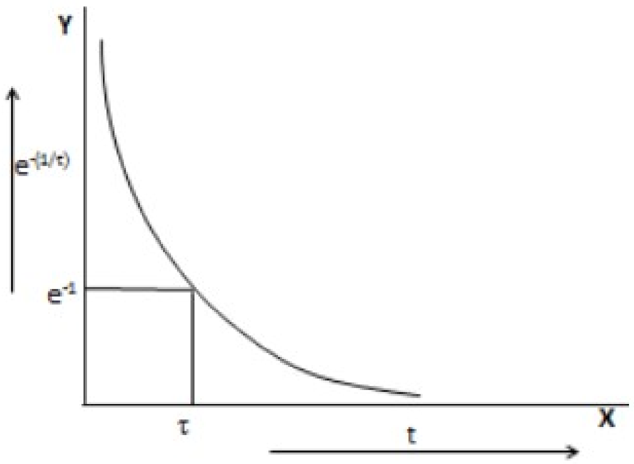


Figure 1

We can express this by saying that velocity is damped by a time constant τ. From the above expression for v, you may note that, at $t = \tau$, $v = v_0 e^{-1} = v_0 / e = v_0 / 2.718 = 0.368 v_0$

Therefore, the time constant, τ (also called relaxation time) may be defined as *the time in which the velocity of the oscillating particle falls to 1/e times (i.e., 0.368 times) of its initial value.*

DIFFERENTIAL EQUATION OF A DAMPED OSCILLATOR

For studying the effect of damping on a one dimensional oscillator, we can consider the representative case of a spring-mass system, as shown in figure below.

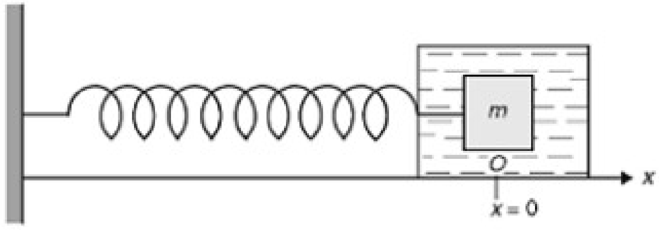


Fig. 2 A damped spring-mass system; the oscillating mass is immersed in a viscous medium

The spring-mass system in which the oscillating mass is executing oscillations in a viscous medium which causes its amplitude progressively decreasing to zero is called a *damped harmonic oscillator*. Obviously, in case of such an oscillator, in addition to the restoring force $-kx$, a resistive or damping force also acts upon it. This damping force is proportional to the velocity, v ($= dx/dt$). We, therefore, can write the equation of the damped spring-mass system as

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - kx$$

or

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \qquad \text{- (5)}$$

This can further be written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \text{- (6)}$$

where $\frac{k}{m} = \omega_0{}^2$ is the natural frequency of oscillating particle (i.e. its frequency in the absence of damping), $\frac{\gamma}{m} = 2b$ (k is the damping constant of the resistive medium)

Above equation is called the differential equation of a damped harmonic oscillator.

6.6 SOLUTION OF THE DIFFERENTIAL EQUATION OF DAMPED HARMONIC OSCILLATOR

The above differential equation is a second order linear homogeneous differential equation. Therefore, it will have at least one solution of type $x = Ae^{\alpha t}$

Here α and t both are arbitrary constants.

Therefore,

$$\frac{dx}{dt} = \alpha A e^{\alpha t} \quad \text{and} \quad \frac{d^2x}{dt^2} = \alpha^2 A e^{\alpha t}$$

Substituting these values in the differential equation (6) above we get

$$\alpha^2 A e^{\alpha t} + 2b\alpha A e^{\alpha t} + \omega_0{}^2 A e^{\alpha t} = 0$$

Or

$$\alpha^2 + 2b\alpha + \omega_0{}^2 = 0 \qquad \text{-------------------------------} (7)$$

This is a quadratic equation in α having its solution of the form

$$\alpha = -b \pm \sqrt{b^2 - \omega_0^2}$$

Thus the original differential equation is satisfied by following two values of x

$$x = Ae^{(-b+\sqrt{b^2-\omega_0^2})t}$$

$$\text{and } x = Ae^{(-b-\sqrt{b^2-\omega_0^2})t}$$

Since the equation being a linear one, the linear sum of two linearly independent solutions will also be a general solution.

Therefore,

$$x = A_1 e^{(-b+\sqrt{b^2-\omega_0^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega_0^2})t} \qquad \text{----------------------} (8)$$

Here A_1 and A_2 are arbitrary constants.

$$\text{Or } \; x = A_1 e^{-\frac{t}{2\tau}+\beta t} + A_2 e^{-\frac{t}{2\tau}-\beta t} \qquad \text{------------------------} (9)$$

where $b = \frac{1}{2\tau}$ and $\beta = \sqrt{b^2 - \omega_0^2}$

The values of the constants A_1 and A_2 can be determined as given below:

Differentiating Eq. (9) with respect to t, we get

$$\frac{dx}{dt} = (-\frac{1}{2\tau} + \beta)A_1 e^{-\frac{t}{2\tau}+\beta t} + (-\frac{1}{2\tau} - \beta)A_2 e^{-\frac{t}{2\tau}-\beta t} \quad \text{----------------- (10)}$$

Now at t=0, displacement must be maximum, i. e. $x_{max}=a_0=A_1+A_2$ and $\frac{dx}{dt} = 0$

Putting t=0 in Eq. (10)

$$(-\frac{1}{2\tau} + \beta)A_1 + (-\frac{1}{2\tau} - \beta)A_2 = 0$$

$$-\frac{1}{2\tau}(A_1 + A_2) + \beta(A_1 - A_2) = 0$$

$$-\frac{1}{2\tau}(a_0) + \beta(A_1 - A_2) = 0$$

$$\beta(A_1 - A_2) = \frac{a_0}{2\tau}$$

$$\text{Or } (A_1 - A_2) = \frac{a_0}{2\tau\beta} \quad \text{--- ---------------- (11)}$$

As we know $A_1 + A_2 = a_0$

Adding it with (11), we get

$$A_1 = \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

$$\text{And } A_1 = (A_1 + A_2) - A_1$$

$$= a_0 - \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

$$= \frac{a_0}{2}\left[1 - \frac{1}{2\tau\beta}\right]$$

Putting these values in equation (9), we get-

$$x = \frac{a_0 e^{-\frac{t}{2\tau}}}{2}\left[\left(1 + \frac{1}{2\tau\beta}\right)e^{\beta t} + \left(1 - \frac{1}{2\tau\beta}\right)e^{-\beta t}\right] \quad \text{---- ------------- (12)}$$

For analysis purpose, above equation may be written as

$$x = \frac{a_0 e^{-\frac{t}{2\tau}}}{2}\left[\left(1 + \frac{1}{2\tau\beta}\right)e^{(\sqrt{b^2-\omega_0^2})t} + \left(1 - \frac{1}{2\tau\beta}\right)e^{-(\sqrt{b^2-\omega_0^2})t}\right] \quad \text{---------- (13)}$$

Now Eq. (13) can be discussed according to following three cases.

6.6.1 CASE I: WHEN b (OR $\frac{1}{2\tau}$)>ω_0, CASE OF OVERDAMPING

In such case $\sqrt{(b^2 - \omega_0^2)}$ is a real quantity, with a positive value. This means that each term in the R. H. S. of Eq. (13), has an exponential term with a negative power.Therefore, the displacement of the oscillator, after attaining a maximum, dies off exponentially with time. Thus, after some time, there will be no oscillations. Such kind of oscillatory motion is called **overdamped** or **aperiodic** motion. Such kind of motion we see in case of dead beat galvanometer.

6.6.2 CASE II: WHEN b (OR $\frac{1}{2\tau}$)= ω_0, CASE OF CRITICAL DAMPING

In such case $\sqrt{(b^2 - \omega_0^2)} = 0$ .Therefore, each term on R. H. S. of Eq. (13) becomes infinite.

Still we can assume that, $\sqrt{(b^2 - \omega_0^2)} = h$ (where h is a very small quantity but not zero obviously).

Therefore Equation (8) gives-

$$x = A_1 e^{(-b+h)t} + A_2 e^{(-b-h)t}$$

$$= e^{-bt}(A_1 e^{ht} + A_2 e^{-ht})$$

$$= e^{-bt}[A_1(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots\ldots\ldots) + A_2(1 - ht + \frac{h^2t^2}{2!} - \frac{h^3t^3}{3!} + \ldots\ldots\ldots)]$$

Neglecting the terms containing higher powers of h, we obtain-

$$x = e^{-bt}[A_1(1 + ht) + A_2(1 - ht)]$$

$$= e^{-bt}[(A_1 + A_2) + (A_1 - A_2)ht]$$

$$= e^{-bt}[M + Nt] \quad \text{------------------------------------- 14)}$$

Here $(A_1 + A_2) = M$ and $(A_1 - A_2)h = N$

Further at $t = 0, x = x_{max} = a_0$

And $\frac{dx}{dt} = 0$

Therefore, the above equation becomes

$$a_0 = M$$

Differentiating Eq. (14), we get

$$\frac{dx}{dt} = \frac{d}{dt}(M\, e^{-bt}) + \frac{d}{dt}(Nte^{-bt})$$

$$0 = -\, b\, M\, e^{-kt} + Ne^{-bt} - Nte^{-bt}$$

$$= -bM + N$$

Or $N = ba_0$

Putting these values of M and N in equation (14) above

$$x = e^{-bt}(a_0 + ba_0t)$$

$$= a_0e^{-bt}(1 + bt)$$

$$= a_0\, e^{-\frac{t}{2\tau}}(1 + \frac{t}{2\tau})$$

$$= a_0\, e^{-\frac{t}{2\tau}} + a_0\, e^{-\frac{t}{2\tau}}(\frac{t}{2\tau})$$

An important feature of the above expression is that its second term decays less rapidly as compared to its first term. In such cases, the displacement of the oscilator first increases, then

quickly return back to its equlibrium position. This kind of oscillatory motion is known as *just aperiodic* (it just ceases to oscillate), or non oscillatory. This case is known as the **critical damping**.

Critical damping finds many applications in many pointer type instruments like, galvenometers, where the pointer moves to and stays at, the correct position, without any further oscillations.

6.6.3 CASE III: WHEN b (OR $\frac{1}{2\tau}$) < ω_0 ,CASE OF WEAK (UNDER) DAMPING

In such cases, the quantity $\sqrt{(b^2 - \omega_0^2)}$ will be imaginary one.

Let $\sqrt{(b^2 - \omega^2)} = i\omega$, where $i = \sqrt{(-1)}$ and $\omega = \sqrt{(\omega_0^2 - b^2)}$ is a real quantity

Putting the values –

$$x = A_1 e^{(-b+iw)t} + A_2 e^{(-b-iw)t}$$

$$= e^{-bt}[A_1(\cos\omega t + i\sin\omega t) + A_2(\cos\omega t - i\sin\omega t)]$$

$$= e^{-bt}[\cos\omega t\,(A_1 + A_2) + \sin\omega t\{i\,(A_1 - A_2)\}]$$

$$= e^{-bt}[A\cos\omega t + B\sin\omega t]$$

where $(A_1 + A_2) = A$ and $i(A_1 - A_2) = B$

$$= e^{-kt}[\,a_0\cos\omega t\,.\,\frac{A}{a_0} + a_0\sin\omega t\,.\,\frac{B}{a_0}\,]$$

Considering a right angle triangle as below in Fig. 3.

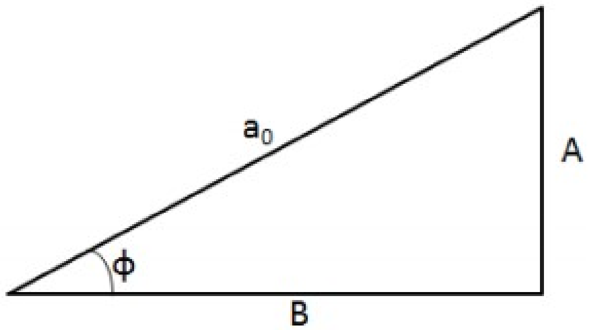


Figure 3

$$\sin\varphi = \frac{A}{a_0},\ \cos\varphi = \frac{B}{a_0}$$

so the above expression can be rewritten as-

$$x = e^{-bt}[\,a_0\{\cos\omega t\,.\,\sin\varphi + \sin\omega t\,.\cos\varphi\}]$$

$$= a_0 e^{-bt}\sin(\omega t + \varphi)$$

$$or\ \ x = a_0 e^{-b\frac{t}{2x}}\sin(\omega t + \varphi)$$

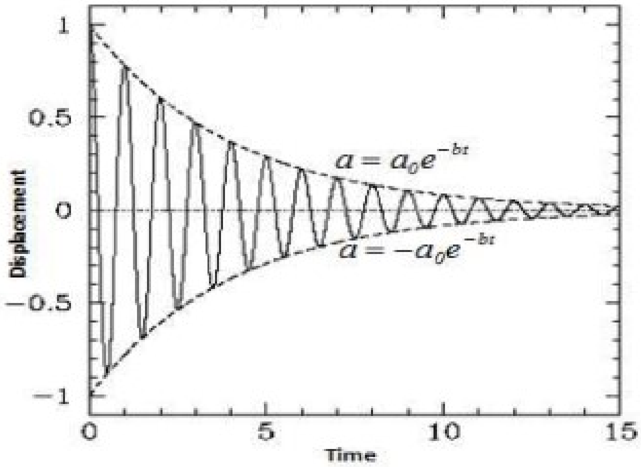
This is the equation of a damped harmonic oscillator with amplitude $a_0 e^{-bt}$ or $x = a_0 e^{-b\frac{t}{2x}}$.

The sine term in the equation suggests that the motion is oscillatory whereas, the exponential term implies that the amplitude is decreasing gradually.

Therefore, we may conclude that the damping produces two effects:

(i) The frequency of damped harmonic oscillator, $\frac{\omega}{2\pi}$ is smaller than its natural frequency $\frac{\omega_0}{2\pi}$, or damping somewhat decreases the frequency or increases the time period of oscillator.

(ii) The amplitude of the oscillator does not remain constant at a_0, which represents amplitude in the absence of damping, but decays exponentially with time, according to the value of term e^{-bt}.



7.3 FORCED DAMPED HARMONIC OSCILLATOR

A damped harmonic oscillator on which an external periodic force is applied is called a forced damped harmonic oscillator. Such an oscillator is also called a driven harmonic oscillator. In such an oscillator, the frequency of the externally applied periodic force is not necessarily the same as the natural frequency of the oscillator. In such a case, there is a sort of tussle between the damping forces tending to retard the motion of the oscillator and the externally applied periodic force which tend to continue the oscillatory motion. As a result, after some initial erratic movements, the oscillator ultimately succumbs to the applied or the driving force and settles down to oscillating with the driving frequency and a constant amplitude and phase so long as the applied force remains operative.

7.3.1 DIFFERENTIAL EQUATION FOR FORCED DAMPED HARMONIC OSCILLATOR

When an external periodic force *F(t)* is applied to a damped harmonic oscillator, the differential equation for the oscillator will have one additional term for the applied time dependent periodic force and we can write

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x + F(t) = 0 \qquad (1)$$

If the applied external force is represented as $F(t) = fcos(nt)$, where *f* and *n* are constants, then Eq. (1) becomes

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f\cos(nt) \qquad (2)$$

We can further simplify the equation as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = \mathrm{a}\cos(nt) \qquad (3)$$

where a = *f*/m, $2b = \gamma$ and $\omega_0^2 = k/m$ is the natural frequency of the oscillator

Eq. (3) represents the differential equation for damped forced harmonic oscillator.

7.3.3 GENERAL SOLUTION

The general solution of the differential equation for damped forced harmonic oscillator comprises of two terms - one representing the homogeneous ordinary differential equation part and the other representing the particular integral part.

$$x(t) = x_H(t) + x_p(t) \qquad (4)$$

$x_H(t)$ is the solution of the corresponding homogeneous part of the equation. The homogeneous part is same as the differential equation for the solution of damped harmonic oscillator and its solution is given as

$$x_H(t) = \frac{1}{2}a_0 e^{-\lambda t}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega_0^2}}\right) e^{\sqrt{(\lambda^2 - \omega_0^2)}t} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \omega_0^2}}\right) e^{-\sqrt{(\lambda^2 - \omega_0^2)}t}\right] \qquad (5)$$

To obtain the particular solution, $x_p(t)$, let us assume the solution of the form

$$x_p(t) = Acos(nt - \emptyset) \qquad (6)$$

Ø is the possible phase difference between the applied force and the displacement of the oscillator and *n* is the frequency of the applied force.

Now we have to obtain *dx/dt* and d^2x/dt^2 and substitute in Eq. (3). We have

$$\frac{dx}{dt} = -Ansin(nt - \emptyset)$$

$$\frac{d^2x}{dt^2} = -An^2cos(nt - \emptyset)$$

Substitution in Eq. (3) gives

$$-An^2\cos(nt - \emptyset) - 2\lambda An\sin(nt - \emptyset) + A\omega_0^2\cos(nt - \emptyset)$$
$$= \text{a}\cos[(nt - \emptyset) + \emptyset] \qquad (7)$$

Expanding the R.H.S gives

$$-An^2\cos(nt - \emptyset) - 2bAn\sin(nt - \emptyset) + A\omega_0^2\cos(nt - \emptyset)$$
$$= a[\cos(nt - \emptyset)\cos\emptyset - \sin(nt - \emptyset)\sin\emptyset \qquad (8)$$

Rearranging we get

$$A(\omega_0^2 - n^2)\cos(nt - \emptyset) - 2bAn\sin(nt - \emptyset) = a[\cos(nt - \emptyset)\cos\emptyset - \sin(nt - \emptyset)\sin\emptyset$$

If this equation is to hold true, then the coefficient of cos(nt – Ø) and sin(nt – Ø) on either sides must be equal

i.e. $A(\omega_0^2 - n^2) = a\cos\emptyset \;\; and \;\; 2bn = a\sin\emptyset$

Squaring and adding these two we get

$$A^2(\omega_0^2 - n^2)^2 + 4b^2n^2 = a^2 \qquad (9)$$

Hence

$$A^2 = \frac{a^2}{(\omega_0^2 - n^2)^2 + 4b^2n^2} \qquad (10)$$

The amplitude of driven or forced oscillator is given as

$$A = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} \qquad (11)$$

We have taken only the positive value of the square root. The negative value will mean opposite phase but then Ø will also change by π and there would, therefore be no effect on the value of *A*. Further, the phase is given by

$$\tan\emptyset = \frac{2bn}{(\omega_0^2 - n^2)} \qquad (12)$$

The particular solution of Eq. (3) is thus given by

$$x_p(t) = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}\cos(nt - \emptyset) \qquad (14)$$

Thus, we can write the general solution as

$$x(t) = \frac{1}{2}a_0 e^{-bt}\left[\left(1+\frac{b}{\sqrt{b^2-\omega_0^2}}\right)e^{\sqrt{(b^2-\omega_0^2)}t} + \left(1-\frac{b}{\sqrt{b^2-\omega_0^2}}\right)e^{-\sqrt{(b^2-\omega_0^2)}t}\right] + \frac{a}{\sqrt{(\omega_0^2-n^2)^2+4b^2n^2}}\cos(nt-\emptyset) \qquad (15)$$

Where $\frac{a_0}{2}$ and Ø need to be determined by initial conditions.

7.3.4 STEADY STATE SOLUTION

When the tussle between the damping and the externally applied forces ends and the oscillator has settled down to oscillate with the frequency of the applied periodic force, it is said to be in the steady state. In the steady state, the homogeneous term vanishes as t → ∞ whereas the particular solution does not. Thus we have a distinction between the transient state , which is a function of the initial conditions, and a steady state , which depends on the external force. Thus, we can write the steady state solution as

$$x(t) = \frac{a}{\sqrt{(\omega_0^2-n^2)^2+4b^2n^2}}\cos(nt-\emptyset) \qquad (16)$$

$$x(t) = A\cos(nt-\emptyset) \qquad (17)$$

$$\text{where } A = \frac{a}{\sqrt{(\omega_0^2-n^2)^2+4b^2n^2}} = \frac{f}{m\sqrt{(\omega_0^2-n^2)^2+4b^2n^2}} \qquad (18)$$

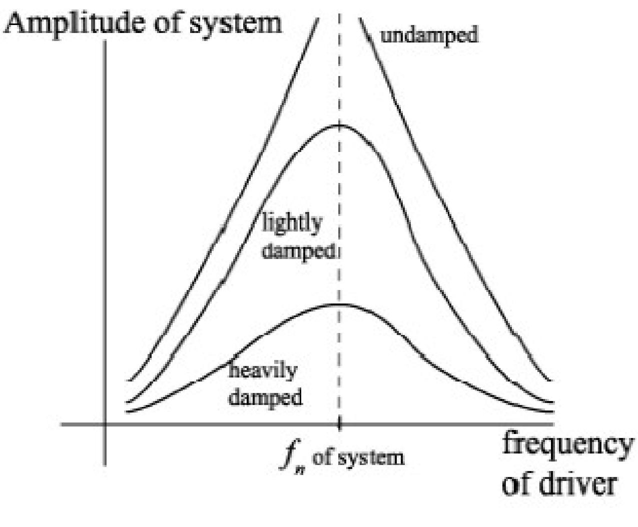


Fig.1: The variation of amplitude of oscillation of a forced oscillator with the frequency of the externally applied periodic force

7.5 RESONANCE

In general, resonance may be defined as a tendency of a vibrating / oscillating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration / oscillation.

For a weakly damped forced (driven) oscillator, after a transitory period, the object will oscillate with the same frequency as that of the driving force. The plot of amplitude $x(\omega)$ versus angular frequency is shown in Fig. 3 below. If the angular frequency is increased from zero, the amplitude, $x(\omega)$ will increase until it reaches a maximum when the angular frequency of the driving force is the same as the natural frequency of the undamped oscillator. This phenomenon is called resonance.

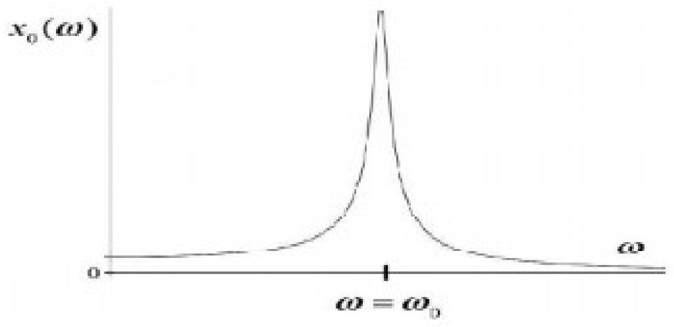


Fig. 3: Plot of amplitude $x(\omega)$ with driving angular frequency ω of a weekly damped harmonic oscillator

From Eq.(18), it is clear that, for a damped forced harmonic oscillator, the amplitude of the oscillator in the steady state depends not only on the amplitude of the driving force, but also on the relation between the frequency, n of the driving force and the natural frequency, ω of the oscillator, as well as on the damping parameter b.

For $n \to 0$ we have $A \to a/\omega$. For $n \to \infty$ we obtain $A \to 0$. In between these two extremes, the amplitude may reach a maximum which we refer to as the resonance frequency.

To obtain an expression for resonance frequency, we differentiate the denominator of Eq. (18) with respect to n and then equate it to zero.

$$\frac{d}{dn}[\omega_0^2 - n^2)^2 + 4b^2n^2] = -4n[\omega^2 - n^2] + 8b^2n = 0$$

The non - trivial solution is:

$$n = n_r = \sqrt{\omega_0^2 - 2b^2} \qquad (26)$$

This is the resonance frequency.

As we have already studied that resonance is defined mathematically using the differential Eq.(2 6) for a forced driven harmonic oscillator where the resonance is defined as the existence of a solution that is unbounded as $t \to \infty$. This corresponds to what we call as pure resonance. It occurs exactly when the natural internal frequency matches the natural external frequency, in which case all solutions of the differential equation are unbounded.

Wave equation in amedium,

Consider a cylindrical metal rod of uniform cross-sectional area. When the rod is struck with a hammer at one end, the disturbance will propagate along it with a speed determined by its physical properties. For simplicity, we assume that the rod is fixed at the left end as shown in Figure 2.

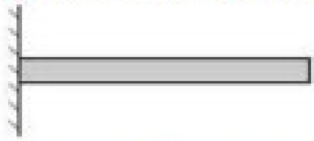


Figure 33: Uniform cylindrical rod fixed at left end.

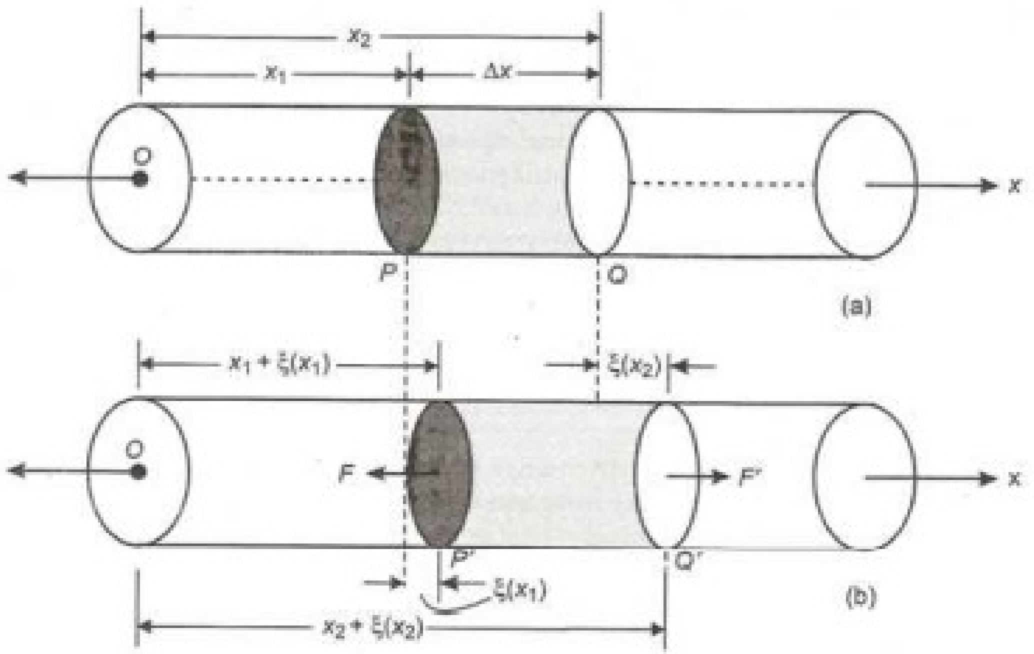


Figure 34: Longitudinal wave propagating in a uniform cylindrical rod. Element PQ in (a) equilibrium state, and (b) deformed state.

We choose x-axis along the length of the rod with origin O at the left end. We divide the rod in a large number of small elements, each of length Δx. Let us consider one such element PQ, as shown in Figure 3a. Since, the rod has been struck at end O lengthwise, the section at P, which is at a distance x_1 from O, will be displaced along x-axis. Since, the force experienced by different sections of the rod is a function of distance, the displacements of particles in different sections will also be function of position. Let us denote is by ξ(x).

Figure 3b shows the deformed state of the rod and displaced position of the element under consideration. Let us denote the x-coordinate of the element in the displaced position by $x_1 + \xi(x_1)$ so that $\xi(x_1)$ represents the displacement of the particles in the section P. Similarly, the new x-coordinate of the particles initially located in the section at Q ($x = x_2$) be denoted by $x_2 + \xi(x_2)$, so that $\xi(x_2)$ signifies the displacement of the particles in section at Q. Hence, the change in length of the element is $\xi(x_2) - \xi(x_1)$. Using Taylor series expansion of $\xi(x_2)$ around x_1 and retaining the first order terms, just like we did in the case of the string, we can write

$$\xi(x_2) - \xi(x_1) = \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \Delta x$$

The linear strain produced in the element PQ can be expressed as

$$\varepsilon(x_2) = \frac{Change\ in\ length}{Original\ length} = \frac{\left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \Delta x}{\Delta x}$$

$$\Rightarrow \varepsilon(x_2) = \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \qquad (10.8)$$

The net force $F' - F$ on the element P'Q' at points P' and Q', as shown in Figure 3b, is toward right. Due to this force, the element under consideration will experience stress, which is the restoring force per unit area. You may recall that the ratio of stress to longitudinal strain defines the Young's modulus Y,

$$Y = \frac{Stress}{Strain}$$

$$\Rightarrow Stress = Y \times Strain$$

In view of the spatial variation of force, we can say that the sections P and Q of the element under consideration will develop different stresses. Therefore, we can write

$$\sigma(x_1) = Y\left(\frac{\partial \xi}{\partial x}\right)_{x=x_1}$$

$$and \qquad \sigma(x_2) = Y\left(\frac{\partial \xi}{\partial x}\right)_{x=x_2}$$

The net stress on the element PQ is

$$\sigma(x_2) - \sigma(x_1) = Y\left[\left(\frac{\partial \xi}{\partial x}\right)_{x=x_2} - \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1}\right]$$

$$= Y[f(x_2) - f(x_1)]$$

where we have put $f(x) = \partial\xi/\partial$x.. As before, using Taylor series expansion for $f(x_2)$ about x_1, we can easily see

$$\sigma(x_2) - \sigma(x_1) = Y\left(\frac{\partial \mathrm{f}}{\partial x}\right)\Delta x$$

$$= Y\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right)\Delta x$$

$$\Rightarrow \sigma(x_2) - \sigma(x_1) = Y\left(\frac{\partial^2 \xi}{\partial x^2}\right)\Delta x \qquad (10.9)$$

If the cross-sectional area of the rod is A, the net force on the elements in the x-direction is given by

$$F(x_2) - F(x_1) = A[\sigma(x_2) - \sigma(x_1)]$$

$$\Rightarrow F(x_2) - F(x_1) = Y\left(\frac{\partial^2 \xi}{\partial x^2}\right)\Delta x \qquad (10.10)$$

Under dynamic equilibrium condition, the equation of motion of the element PQ, using Newton's second law of motion, can be written as

$$Y\left(\frac{\partial^2 \xi}{\partial x^2}\right)\Delta x = \rho A \Delta x\left(\frac{\partial^2 \xi}{\partial t^2}\right) \qquad (10.11)$$

where ρ is the density of the material of the rod and $\rho A\Delta x$ signifies the mass of the element PQ. On simplification, we find that the displacement $\xi(\mathrm{x}, \mathrm{t})$ satisfies the equation

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2 \xi}{\partial x^2} \qquad (10.12)$$

which is of the form of wave equation (10.7) with

$$v = \sqrt{\frac{Y}{\rho}} \qquad (10.13)$$

Equations (10.12) and (10.13) show that the deformation propagates along the rod as a wave and the velocity of the longitudinal waves is independent of the cross-sectional area of the rod.

Velocity of Longitudinal waves in an elastic medium

In order to understand the propagation of one-dimensional longitudinal waves in a gas, consider a gas column in a long pipe or cylindrical tube of uniform cross-sectional area A. As before, we conveniently choose x-axis along the length of the tube and divide the column of the gas into small elements or slices, each of small length Δx. Figure 4 shows one such volume element PQRS. Thus, the volume of this element is V = AΔx.

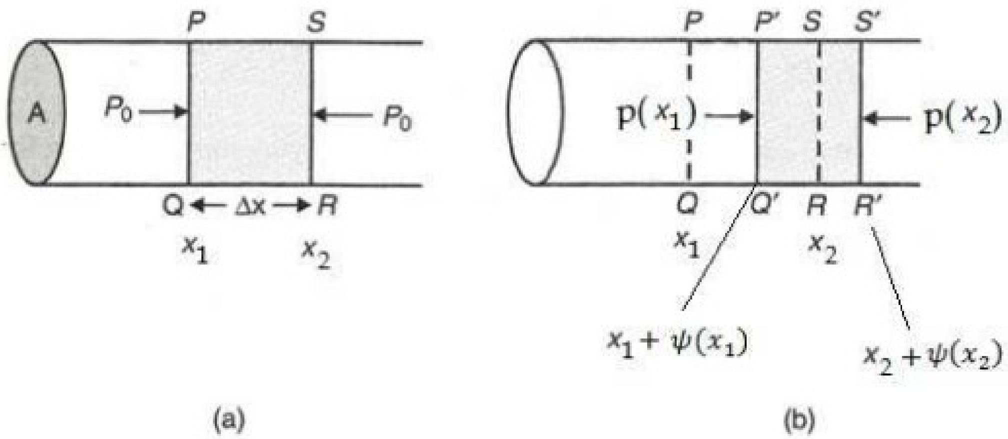


Figure 35: (a) Equilibrium state of the column PQRS of a gas contained in a long tube of cross-sectional area A, and (b) displaced position of column under pressure difference.

Under equilibrium condition, pressure and density of the gas remains the same throughout the volume of the gas, independent of the x-coordinate. Let the equilibrium pressure be denoted by p_0. If the pressure of the gas in the tube is changed, the volume element PQRS will be set in motion giving rise to a net force. Let us choose the origin of the coordinate system so that the particles in plane PQ are at a distance x_1 and those in plane SR are at a distance x_2 from it. Figure 4b shows the displaced position of the volume element when PQ is shifted to P'Q' and SR is shifted to S'R' Let the new coordinates be denoted by $x_1 + \psi(x_1)$ and $x_2 + \psi(x_2)$, respectively. It means that $\psi(x_1)$ and $\psi(x_2)$ respectively, denote the displacements of the particles originally at x_1 and x_2. Therefore, the change in thickness Δl is given by

$$\Delta l = \psi(x_2) - \psi(x_1)$$

If Δl is positive, there is increase in length, and hence the volume of the element also increases and vice versa. Using Taylor series expansion for $\psi(x_2)$ about $\psi(x_1)$, we can write

$$\Delta l = \psi(x_2) - \psi(x_1) = \left(\frac{\partial \psi}{\partial x}\right) \Delta x$$

This means that the change in volume ΔV is

$$\Delta V = A \Delta l = A \Delta x \left(\frac{\partial \psi}{\partial x}\right)$$

The volume strain, which is defined as the change in volume per unit volume, is given by

$$\frac{\Delta V}{V} = \frac{A \Delta x}{A \Delta x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial \psi}{\partial x} \qquad (10.14)$$

This increase in volume of the element is due to the decrease in pressure and vice versa.

It should be noted that until now all the steps that have been followed are identical to the case of the solid rod. However, as mentioned earlier, due to comparatively large compressibility of the gas, change in volume is accompanied by changes in density. This implies that the pressure in the compressed/rarefied gas varies with distance. To proceed further, let us suppose that the pressure at P'Q' is $p_0 + p(x_1')$. Hence, the pressure difference across the ends of the element P'Q'R'S' can be expressed in terms of the pressure gradient,

$$p(x_2') - p(x_1') = \left(\frac{\partial p(x)}{\partial x}\right)_{x = x_1'} \Delta x$$

$$= \frac{\partial (p_0 - \Delta p)}{\partial x} \Delta x$$

Since, p_0 is a constant

$$\Rightarrow p(x_2') - p(x_1') = -\frac{\partial (\Delta p)}{\partial x} \Delta x \qquad (10.15)$$

To express the above result in a familiar form, we note that Δp is connected to the bulk modulus of elasticity by the relation

$$E = \frac{Stress}{Volume\ Strain} = -\frac{\Delta p}{\Delta V / V}$$

The negative sign is included to account for the fact that when the pressure increases, the volume decreases. This ensures that E is positive. We can write the above relation as

$$\Delta p = -E\left(\frac{\Delta V}{V}\right)$$

On substituting for $\Delta V/V$ from equation (10.14), we get

$$\Delta p = -E\left(\frac{\partial \psi}{\partial x}\right)$$

Using equation (10.15), we find that the pressure difference at the ends of the displaced column is given by

$$p(x_2') - p(x_1') = -\frac{\partial}{\partial x}\left(-E\frac{\partial \psi}{\partial x}\right)\Delta x = E\left(\frac{\partial^2 \psi}{\partial x^2}\right)\Delta x$$

The net force acting on the volume element is obtained by multiplying this expression for pressure difference by the cross-sectional are of the column,

$$F = [p(x_2') - p(x_1')]A$$

$$= EA\Delta x\left(\frac{\partial^2 \psi}{\partial x^2}\right)$$

Under the action of this force, the volume element under consideration shall be set in motion. Using Newton's second law of motion, we find that the equation of motion of the element under consideration can be expressed as

$$\rho \Delta x A\frac{\partial^2 \psi}{\partial t^2} = EA\Delta x\left(\frac{\partial^2 \psi}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \psi}{\partial x^2} \tag{10.16}$$

If we identify the speed of the longitudinal wave as

$$v = \sqrt{\frac{E}{\rho}} \tag{10.17}$$

equation (10.16) becomes identical to equation (10.7). One must note that the wave speed is determined only by the bulk modulus of elasticity and density – two properties of the medium through which the wave is propagating.

equation (10.16) becomes identical to equation (10.7). One must note that the wave speed is determined only by the bulk modulus of elasticity and density – two properties of the medium through which the wave is propagating.

When a longitudinal wave propagates through a gaseous medium such as air, the volume elasticity is influenced by the thermodynamic changes that take place in it. These changes can be isothermal or adiabatic. Newton gave the first theoretical expression of the velocity of sound wave in a gas. He assumed that when sound wave travels through a gaseous medium, the temperature variations in the regions of compression and rarefaction are negligible. For sound waves propagating in air, Newton assumed that isothermal changes take place in the medium. For an isothermal change, the volume elasticity equals atmospheric pressure,

$$E = E_T = p$$

Then we can write,

$$v = \sqrt{\frac{p}{\rho}} \tag{10.18}$$

This is known as the Newton's formula for velocity of sound. For air at STP, $\rho = 1.29\ \mathrm{kgm^{-3}}$ and $\mathrm{p} = 1.01 \times 10^5\ \mathrm{Nm^{-2}}$. Hence, velocity of sound in air at STP, using the Newton's formula comes out to be

$$v = \sqrt{\frac{1.01 \times 10^5\ Nm^{-2}}{1.29\ kgm^{-3}}} = 280\ m/s$$

But experimental results paint a different picture and show that the speed of sound in air at STP is actually around 332 m/s, which is about 15% higher than the value predicted by Newton's formula. This implies that something was wrong with the assumption of isothermal change.

For an adiabatic change, E_s is γ times the pressure, where γ is the ratio of specific heat capacities of a gas at constant pressure and at constant volume, i.e.

$$E_s = \gamma \mathrm{p}$$

Then, equation (10.18) becomes

$$v = \sqrt{\frac{\gamma p}{\rho}} \tag{10.19}$$

This is known as the Laplace's formula. For air, $\gamma = 1.4$ and the velocity of sound in air at STP based on equation (10.19) comes out to be 331 m/s, which is in close agreement with the experimentally measured value, thereby establishing the correctness of Laplace's explanation.

At a given temperature, p/ρ is constant for a gas. So, equation (10.19) shows that the velocity of a longitudinal wave is independent of pressure.

Waves on a Stretched String

Consider a uniform stretched string, having mass per unit length m. Under equilibrium conditions, it can be considered to be straight. The x-axis is chosen along the length of the stretched string in its equilibrium state. Let the string be displaced perpendicular to its length by a small amount so that a small section of length Δx is displaced through a distance y from its mean position, as shown in Figure 1. When the string is released, it results in wave motion. Let's see how.

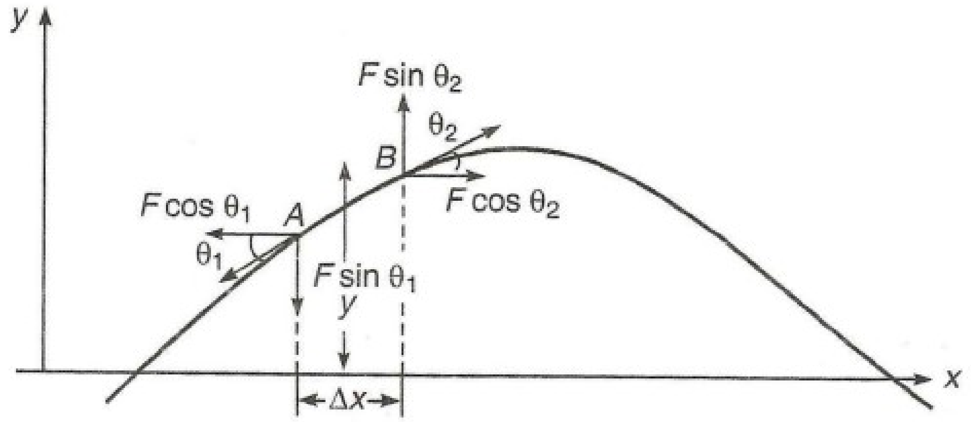


Figure 32: Forces acting on a small element of a string displaced perpendicular to its length.

We have studied that the wave disturbance travels from one particle to another due to their masses (or inertia) and the factor responsible for the periodic motion of the particle is the elasticity of the medium. For a stretched string, the elasticity is measured by the tension F in it and the inertia is measured by mass per unit length or linear mass density, m.

Suppose that the tangential force on each end of a small element AB, as shown in Figure 1, is F; the force on the end B is produced by the pull of the string to the right and the one at A is due to the pull of the string to the left. Due to the curvature of the element AB, the forces are not directly opposite to each other. Instead, they make angles θ_1 and θ_2 with the x-axis. This means that the forces pulling the element AB at opposite ends, though of equal magnitude, do not exactly cancel each other. In order to calculate the net force along the x- and y-axes, the forces are resolved into rectangular components. The net force in the x and the y directions are respectively given by

$$F_x = F\cos\theta_2 - F\cos\theta_1$$

$$and \quad F_y = F\sin\theta_2 - F\sin\theta_1$$

For small angle approximation, $\cos\theta \approx 1$ and $\sin\theta \approx \theta \approx \tan\theta$. This implies that if the displacement of the string perpendicular to its length is relatively small, the angles θ_1 and θ_2 will be small and there is no net force in the x-direction, and the element AB is only subjected to a net upward force F_y. Under the action of this force, the string element will move up and down. Therefore, the y-component of the force on element AB can be written as

$$F_y = F\tan\theta_2 - F\tan\theta_1$$

$$F_y = F\left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_x\right) \tag{10.1}$$

Note that the perpendicular displacement y(x, t) of the string is both a function of the position x and time t. However, equation (10.1) is valid at a particular instant of time. Therefore, the derivative in this expression should be taken by keeping the time fixed. Therefore, equation (10.1) can be rewritten as

$$F_y = F\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x\right) \tag{10.2}$$

For the sake of convenience, let us put

$$f(x) = \left.\frac{\partial y}{\partial x}\right|_x \qquad and \qquad f(x+\Delta x) = \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$$

in equation (10.2). Thus, equation (10.2) becomes

$$F_y = F[f(x+\Delta x) - f(x)] \tag{10.3}$$

To simplify the above expression, we make use of Taylor series expansion of the function $f(x+\Delta x)$ about the point x:

$$f(x+\Delta x) = f(x) + \left.\frac{\partial f}{\partial x}\right|_x \Delta x + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_x \Delta x^2 + \cdots$$

Since, Δx is small , we can ignore the second and the higher order terms in Δx to obtain,

Since, Δx is small , we can ignore the second and the higher order terms in Δx to obtain,

$$f(x+\Delta x) = f(x) + \left.\frac{\partial f}{\partial x}\right|_x \Delta x$$

$$= f(x) + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Delta x$$

$$\Rightarrow f(x+\Delta x) - f(x) = \frac{\partial^2 y}{\partial x^2}\Delta x$$

Inserting the above result in equation (10.3), we get

$$F_y = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

This equation gives the net force on the element AB. We use Newton's second law of motion to obtain the equation of motion of this element, by equating this force to the product of mass and acceleration of the element AB. The mass of the element AB is $m\,\Delta x$. Therefore, we can write

$$m\,\Delta x \frac{\partial^2 y}{\partial t^2} = F \frac{\partial^2 y}{\partial x^2} \Delta x$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{m}{F}\frac{\partial^2 y}{\partial t^2} \tag{10.4}$$

Note that even though equation (10.4) has been obtained for a small element AB, it can be applied to the entire string, since there is nothing special about this particular element of the string. In other words, equation (10.4) can be applied to all the elements of the string.

Now, let us go back to the sinusoidal wave propagating on the string described by the equation

$$y(x,t) = A\sin(\omega t - kx)$$

If this mathematical form is consistent with equation (10.4), then we can be sure that such a wave can indeed move on the string. To check this, we calculate the spatial and the temporal partial derivatives of particle displacement y(x, t):

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(\omega t - kx)$$

$$and \quad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$$

Substituting these partial derivatives in equation (10.4), we get

$$-k^2 A\sin(\omega t - kx) = \frac{m}{F}[-\omega^2 A\sin(\omega t - kx)]$$

$$\Rightarrow \frac{F}{m} = \left(\frac{\omega}{k}\right)^2 \tag{10.5}$$

But, we know that ω/k is the wave speed v, therefore, from the above relation, we get

$$v = \frac{\omega}{k} = \sqrt{\frac{F}{m}} \tag{10.6}$$

The above relation tells us that velocity of a transverse wave on a stretched string depends on tension and mass per unit length of the string. Using equation (10.6), we can write equation (10.4) as

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{10.7}$$

This result expresses one-dimensional wave equation. It holds as long as we deal with small amplitude waves. Elasticity provides the restoring force and the inertia determines the response of the medium.

Lissajous figures for
superposition of two orthogonal simple harmonic vibrations (a) with same frequency

Let's now study a simpler case, where we assume that two independent forces are acting on a particle in such a manner that the first alone produces a simple harmonic motion in the x-direction given by

$$x = A_1 \sin \omega t \tag{5.1}$$

and the second would produce a simple harmonic motion in the y-direction given by

$$y = A_2 \sin(\omega t + \delta) \tag{5.2}$$

Thus, we are actually considering the superposition of two mutually perpendicular SHMs which have equal frequencies. The amplitudes may be different and their phases differ by δ. The resultant motion of the particle is a combination of the two SHMs.

The position of the particle at any time t is given by (x, y) where x and y are given by the above equations. The *resultant motion is, thus, two-dimensional* and the path of the particle is, in general, an ellipse. The equation of the path traced by the particle is obtained by eliminating t from equations (5.1) and (5.2).

From equation (5.1), we get

$$\sin \omega t = \frac{x}{A_1} \text{ ; which gives } \quad \cos \omega t = \sqrt{1 - \left(\frac{x}{A_1}\right)^2}$$

Putting these values in equation (5.2), we get

$$y = A_2 \sin(\omega t + \delta) = A_2[\sin \omega t \cos \delta + \cos \omega t \sin \delta]$$

$$= A_2 \left[\left(\frac{x}{A_1}\right) \cos \delta + \left(\sqrt{1 - \left(\frac{x}{A_1}\right)^2}\right) \sin \delta\right]$$

Or,

$$\left(\frac{y}{A_2} - \frac{x}{A_1} \cos \delta\right)^2 = \left(1 - \left(\frac{x}{A_1}\right)^2\right) \sin^2 \delta$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy \cos \delta}{A_1 A_2} = \sin^2 \delta \tag{5.3}$$

As we can see, equation (5.3) is an equation of ellipse. Thus, we may conclude that the resultant motion of the particle is along an elliptical path.

Equation (5.3) shows that x remains between $-A_1$ and A_1 and that of y remains between $-A_2$ and A_2. Thus, the particle always remains inside the rectangle defined by

$$x = \pm A_1 \quad and \quad y = \pm A_2$$

The ellipse given by equation (5.3) is shown in the figure below:

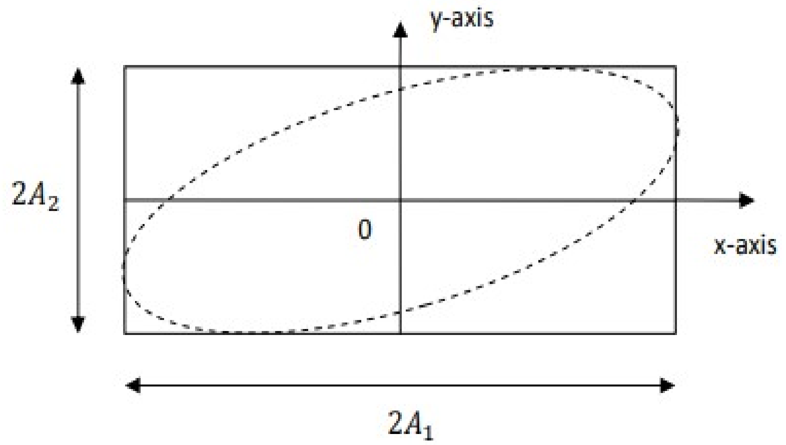


Figure 23: Elliptical path followed by a particle on which two independent SHMs, which are perpendicular to each other, act simultaneously.

<u>*Special Cases*</u>

- The two component SHMs are in phase, $\delta = 0$
- The two component SHMs are out of phase, $\delta = \pi$
- The phase difference between the two component SHMs, $\delta = \pi/2$

Let us now obtain the resultant motion of the particle under thes special cases.

(a) When the two superposing SHMs are in phase, $\delta = 0$ and equation (5.3) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} - \frac{x}{A_1}\right)^2 = 0$$

$$\therefore y = \frac{A_2}{A_1} x \qquad (5.4)$$

Equation (5.4) is an equation of a straight line passing through the origin and having a slope of $\tan^{-1}\left(\frac{A_2}{A_1}\right)$. The figure below shows the path followed by the particle in this case. The particle moves on the diagonal (shown by the dotted line) of the rectangle.

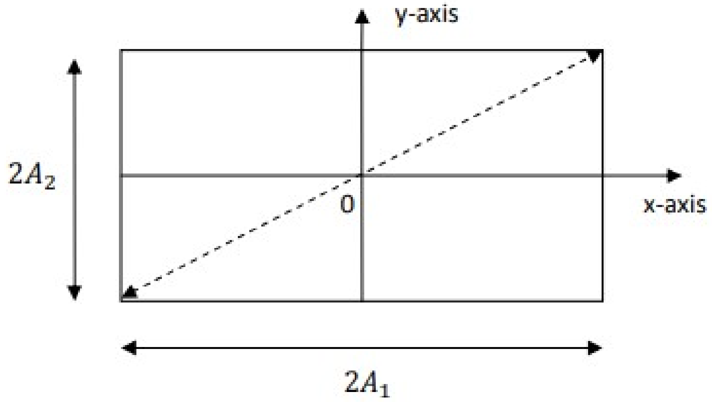


Figure 24: The straight line path traced by the resultant motion of the particle when the phase difference, $\delta = 0$.

Equation (5.4) can also be obtained directly from equations (5.1) and (5.2) putting $\delta = 0$. The displacement of the particle on this straight line at any time t is

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(A_1 \sin \omega t)^2 + (A_2 \sin \omega t)^2} = \sqrt{A_1^2 + A_2^2} \sin \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency and phase as the component motions. However, the amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

(b) When the two superposing SHMs are out of phase, the phase difference between them is $\delta = \pi$. Thus, from equation (5.3), we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} + \frac{x}{A_1}\right)^2 = 0$$

$$\therefore y = -\frac{A_2}{A_1} x \qquad (5.5)$$

Equation (5.5) is an equation of a straight line passing through the origin and having a slope $\tan^{-1}\left(-\frac{A_2}{A_1}\right)$. The figure below shows the path followed by the particle. The particle moves on one of the diagonals (shown by dotted line) of the rectangle.

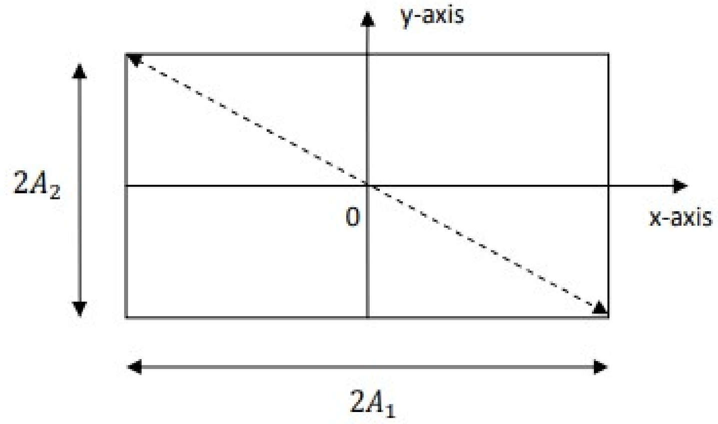


Figure 4: The straight line path traced by the resultant motion of the particle when the phase difference, $\delta = \pi$.

Equation (5.5) can also be obtained directly on the basis of equations (5.1) and (5.2) and putting $\delta = \pi$. Further, the displacement of the particle on this straight line path at a given time t is

$$r = \sqrt{x^2 + y^2} = \sqrt{(A_1 \sin \omega t)^2 + (A_2 \sin(\omega t + \pi))^2}$$

$$= \sqrt{(A_1 \sin \omega t)^2 + (-A_2 \sin \omega t)^2} = \sqrt{A_1^2 + A_2^2} \sin \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency as the component motions. The amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

(c) When the phase difference between the two component SHMs is $\delta = \pi/2$.

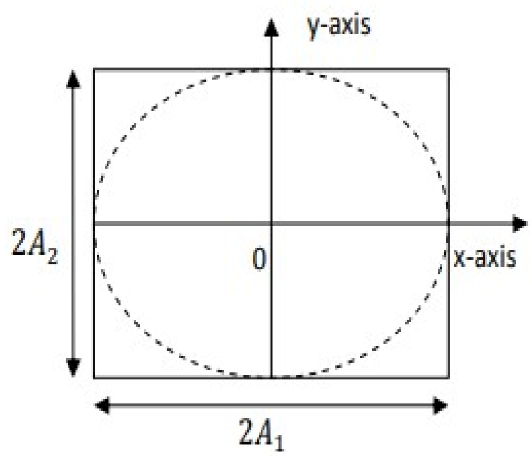


Figure 5: The elliptical path traced by the resultant motion of the particle when the phase difference, $\delta = \pi/2$.

From equation (5.3), we get

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \tag{5.6}$$

The above equation is a standard equation of an ellipse with its axes along the x-axis and the y-axis and with its center at the origin. The lengths of the major and the minor axes are $2A_1$ and $2A_2$, respectively. The path traced by the particle (shown by the dotted line) is depicted in Fig. 5.

In case the amplitudes of the two individual SHMs are equal, $A_1 = A_2 = A$, i.e. the major and the minor axes are equal, then the ellipse reduces to a circle.

$$x^2 + y^2 = A^2 \tag{5.7}$$

Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of $\pi/2$ is a *circular motion*. The circular motion may be clockwise or anticlockwise depending on which component leads the other.

Example 1: Show that the superposition of oscillations represented by

$$x = A \sin \omega t$$

$$y = -A \cos \omega t$$

results in to circular motion traced in the anticlockwise sense.

3.2. SUPERPOSITION OF TWO PERPENDICULAR SIMPLE HARMONIC MOTIONS OF DIFFERENT AMPLITUDES AND FREQUENCIES IN THE RATIO 2 : 1

Analytical Method

Let a particle is subjected to two mutually perpendicular simple harmonic vibrations (S.H.M.) having frequencies in the ratio 2 : 1, different amplitudes and phase difference.

They may be represented by

$$x = a \cos (2\omega t + \theta) \quad \text{...(3.6)}$$

and

$$y = b \cos \omega t \quad \text{...(3.7)}$$

where 'a' is the amplitude of S.H.M. along X-axis and angular frequency 2ω and 'b' is the amplitude of S.H.M. along Y-axis with frequency ω. The phase difference is θ.

From eq. (3.6),

$$\frac{x}{a} = \cos (2\omega t + \theta)$$

$$\frac{x}{a} = \cos 2\omega t \cos \theta - \sin 2\omega t \sin \theta$$

$$\frac{x}{a} = (2\cos^2 \omega t - 1)\cos\theta - 2\sin\omega t\cos\omega t\sin\theta \quad ...(3.8)$$

From eq. (3.7), $\frac{y}{b} = \cos\omega t$

Putting this value in eq. (3.8),

$$\frac{x}{a} = \left(\frac{2y^2}{b^2} - 1\right)\cos\theta - 2\sin\omega t\left(\frac{y}{b}\right)\sin\theta$$

$$\frac{x}{a} = \left(\frac{2y^2}{b^2} - 1\right)\cos\theta - \frac{2y}{b}\sin\omega t\sin\theta$$

$$\frac{x}{a} = \left(\frac{2y^2}{b^2} - 1\right)\cos\theta - \frac{2y}{b}\sqrt{\sin^2\omega t}\sin\theta$$

$$\frac{x}{a} = \left(\frac{2y^2}{b^2} - 1\right)\cos\theta - \frac{2y}{b}\sqrt{1-\cos^2\omega t}\sin\theta$$

$$\frac{x}{a} = \left(\frac{2y^2}{b^2} - 1\right)\cos\theta - \frac{2y}{b}\sqrt{1-\frac{y^2}{b^2}}\sin\theta$$

$$\frac{x}{a} - \left(\frac{2y^2}{b^2} - 1\right)\cos\theta = -\frac{2y}{b}\sqrt{1-\frac{y^2}{b^2}}\sin\theta$$

Squaring both sides,

$$\left[\frac{x}{a} - \left(\frac{2y^2}{b^2} - 1\right)\cos\theta\right]^2 = \left[-\frac{2y}{b}\sqrt{1-\frac{y^2}{b^2}}\sin\theta\right]^2$$

$$\frac{x^2}{a^2} + \left(\frac{2y^2}{b^2} - 1\right)^2\cos^2\theta - \frac{2x}{a}\left(\frac{2y^2}{b^2} - 1\right)\cos\theta = \frac{4y^2}{b^2}\left(1-\frac{y^2}{b^2}\right)\sin^2\theta$$

$$\frac{x^2}{a^2} + \left(\frac{4y^4}{b^4} + 1 - \frac{4y^2}{b^2}\right)\cos^2\theta - \frac{4xy^2}{ab^2}\cos\theta + \frac{2x}{a}\cos\theta = \frac{4y^2}{b^2}\sin^2\theta - \frac{4y^4}{b^4}\sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{4y^4}{b^4}\cos^2\theta + \cos^2\theta - \frac{4y^2}{b^2}\cos^2\theta - \frac{4xy^2}{ab^2}\cos\theta + \frac{2x}{a}\cos\theta = \frac{4y^2}{b^2}\sin^2\theta - \frac{4y^4}{b^4}\sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{4y^4}{b^4}\cos^2\theta + \frac{4y^4}{b^4}\sin^2\theta - \frac{4y^2}{b^2}\sin^2\theta - \frac{4y^2}{b^2}\cos^2\theta - \frac{4xy}{ab^2}\cos\theta + \cos^2\theta + \frac{2x}{a}\cos\theta = 0$$

$$\frac{x^2}{a^2} + \frac{4y^4}{b^4}(\sin^2\theta + \cos^2\theta) - \frac{4y^2}{b^2}(\sin^2\theta + \cos^2\theta) - \frac{4xy^2}{ab^2}\cos\theta + \cos^2\theta + \frac{2x}{a}\cos\theta = 0$$

$$\frac{x^2}{a^2} + \frac{4y^4}{b^4}(1) - \frac{4y^2}{b^2}(1) - \frac{4xy^2}{ab^2}\cos\theta + \cos^2\theta + \frac{2x}{a}\cos\theta = 0$$

$$\frac{x^2}{a^2} + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} - \frac{4xy^2}{ab^2}\cos\theta + \cos^2\theta + \frac{2x}{a}\cos\theta = 0$$

$$\left(\frac{x^2}{a^2}+\cos^2\theta+\frac{2x}{a}\cos\theta\right)+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-\frac{x}{a}\cos\theta-1\right)=0$$

$$\left(\frac{x}{a}+\cos\theta\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-\frac{x}{a}\cos\theta-1\right)=0 \qquad ...(3.9)$$

This is the general equation of a curve having two loops.

The curve is described within a rectangle of sides $2a$ and $2b$ for all the values of phase difference θ.

Special Cases :

Case Ist : When $\theta = 0$ or 2π

i.e. two component of vibrations are in phase, then

$$\cos\theta = 1$$

Putting this value in eq. (3.9)

$$\left(\frac{x}{a}+1\right)^2+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-\frac{x}{a}-1\right)=0$$

$$\left(\frac{x}{a}+1\right)^2+\frac{4y^4}{b^4}-\frac{4y^2}{b^2}\left(\frac{x}{a}+1\right)=0$$

$$\left[\left(\frac{x}{a}+1\right)-\frac{2y^2}{b^2}\right]^2=0$$

$$\left(\frac{x}{a}+1\right)-\frac{2y^2}{b^2}=0$$

$$\frac{2y^2}{b^2}=\left(\frac{x}{a}+1\right)$$

$$y^2=\frac{b^2}{2}\left(\frac{x}{a}+1\right)$$

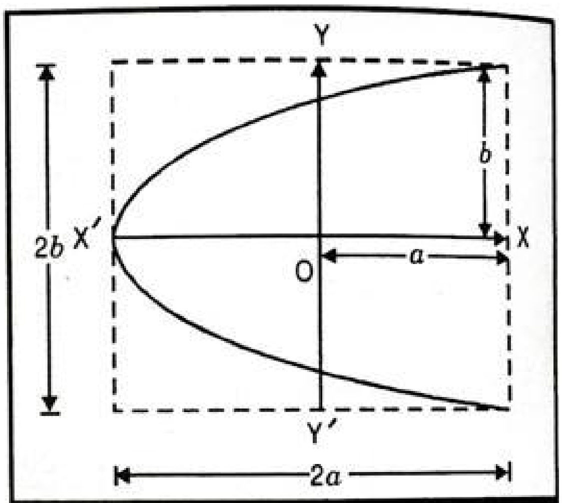


Fig. 3.7.

This is the equation of a parabola symmetrical about X-axis as shown in Fig. 3.7.

Case IInd :

When $\theta = \frac{\pi}{2}$,

then $\cos\theta = \cos\frac{\pi}{2} = 0$

Putting this value in eq. (3.9)

$$\frac{x^2}{a^2}+\frac{4y^2}{b^2}\left(\frac{y^2}{b^2}-1\right)=0$$

This equation represents the figure 8 as shown in Fig. (3.8)

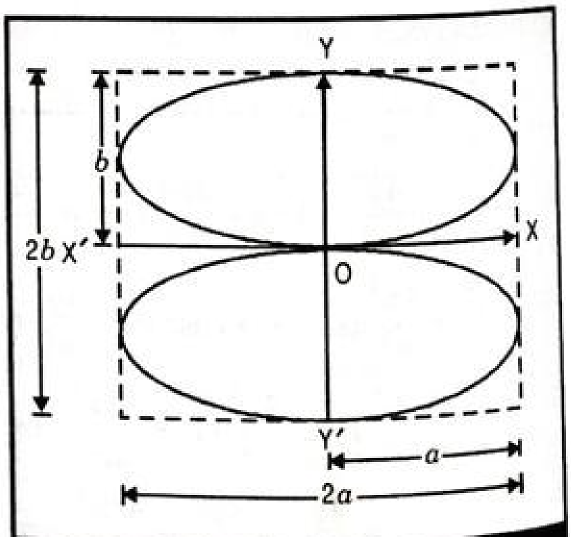


Fig. 3.8.

Case IIIrd :

When $\theta = \pi$, then $\cos\theta = \cos\pi = -1$

Putting this value in eq. (3.9)

$$\left(\frac{x}{a}-1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2}+\frac{x}{a}-1\right) = 0$$

$$\left(\frac{x}{a}-1\right)^2 + \frac{4y^4}{b^4} + \frac{4y^2}{b^2}\left(\frac{x}{a}-1\right) = 0$$

$$\left[\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2}\right]^2 = 0$$

$$\left(\frac{x}{a}-1\right)+\frac{2y^2}{b^2} = 0$$

$$\frac{2y^2}{b^2} = \left(1-\frac{x}{a}\right)$$

$$y^2 = \frac{b^2}{2}\left(1-\frac{x}{a}\right)$$

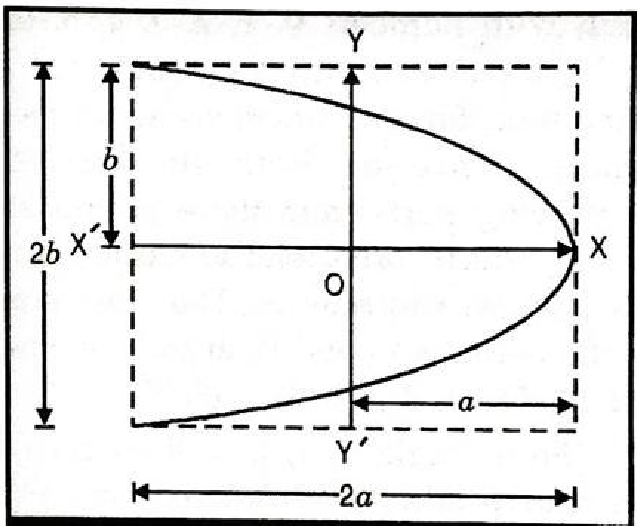


Fig. 3.9.

This is the equation of a parabola symmetrical about X-axis as shown in Fig. (3.9).